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NOETHER SKEW SEMI RESIDUATED ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT

In this paper, we introduce and study the concept of Noether Skew Semi-Residuated Almost Distributive Lattices (Noether SSR ADLs). We define and examine the notion of principal elements within a Skew Semi-Residuated Almost Distributive Lattice (SSR ADL) and establish several fundamental properties of these structures. Furthermore, we develop and prove significant results concerning the algebraic behavior of Noether SSR ADLs, culminating in the formulation and proof of the Fundamental Theorem of Noether SSR ADLs. These results provide new insights into the structural and order-theoretic characteristics of residuated lattice systems with skew and almost distributive properties.

Introduction

Swamy, U.M. and Rao, G.C. [6] introduced the concept of an Almost Distributive Lattice (ADL) as a unifying abstraction encompassing nearly all ring-theoretic generalizations of Boolean algebras such as regular rings, p-rings, biregular rings, associate rings, and P_1-rings on one hand, and distributive lattices on the other. In [1], Dilworth, R.P. introduced the notion of residuation in lattices, and later, Ward, M. and Dilworth, R.P. [7] studied residuated lattices in detail.

In our earlier work [2], we introduced the concepts of skew semi-residuation and multiplication in an ADL, thereby defining a skew semi-residuated ADL. We also established several fundamental properties of skew semi-residuation (denoted by '$\circ$') and multiplication (denoted by'$\cdot$') within such lattices.

In this paper, we extend that study by introducing the notions of a Noether Skew Semi-Residuated Almost Distributive Lattice (Noether SSR-ADL) and principal elements in a Skew Semi-Residuated Almost Distributive Lattice (SSR-ADL). Furthermore, we establish several significant results pertaining to Noether SSR-ADLs.

In Section 2, we revisit the definition of an Almost Distributive Lattice (ADL) and some fundamental properties of ADLs as presented by Swamy, U.M. and Rao, G.C. [6], and Rao, G.C. [3]. We also recall the concepts of skew semi-residuation and multiplication in an ADL LLL, along with the definition of a Skew Semi-Residuated Almost Distributive Lattice (SSR-ADL) from our earlier work [2].

In **Section 3**, we introduce the notions of a Noether Skew Semi-Residuated Almost Distributive Lattice (Noether SSR-ADL) and principal elements in an SSR-ADL.

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We introduce the concepts of a Noether Skew Semi-Residuated Almost Distributive Lattice (Noether SSR-ADL) and principal elements in a Skew Semi-Residuated Almost Distributive Lattice (SSR-ADL). Furthermore, we establish several important results in a Noether SSR-ADL and prove the Fundamental Theorem of Noether SSR-ADLs.

Preliminaries

In this section, we present several important definitions and known results that will be used frequently throughout the paper.

We begin with the definition of an **Almost Distributive Lattice (ADL):**

Definition 2.1 ([3]).

An Almost Distributive Lattice (ADL) is an algebra (L,∨,∧)(L, \vee, \wedge)(L,∨,∧) of type (2,2)(2, 2)(2,2) satisfying the following conditions for all a,b,c∈La, b, c \in La,b,c∈L:

(a∨b)∧c=(a∧c)∨(b∧c)(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)(a∨b)∧c=(a∧c)∨(b∧c)

a∧(b∨c)=(a∧b)∨(a∧c)a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)a∧(b∨c)=(a∧b)∨(a∧c)

1. (a∨b)∧b=b(a \vee b) \wedge b = b(a∨b)∧b=b
2. (a∨b)∧a=a(a \vee b) \wedge a = a(a∨b)∧a=a
3. a∨(a∧b)=aa \vee (a \wedge b) = aa∨(a∧b)=a

It is evident that every distributive lattice is an ADL. Moreover, if there exists an element 0∈L0 \in L0∈L such that 0∧a=00 \wedge a = 00∧a=0 for all a∈La \in La∈L, then (L,∨,∧,0)(L, \vee, \wedge, 0)(L,∨,∧,0) is called an ADL with 0.

Example 2.1 ([3]).

Let XXX be a non-empty set, and fix an element x0∈Xx_0 \in Xx0∈X. For any x,y∈Xx, y \in Xx,y∈X, define the operations ∧\wedge∧ and ∨\vee∨ as follows:

x∧y={x0,if x=x0,y,if x≠x0,x∨y={y,if x=x0,x,if x≠x0.x \wedge y = \begin{cases} x_0, & \text{if } x = x_0,\\ y, & \text{if } x \neq x_0, \end{cases} \qquad x \vee y = \begin{cases} y, & \text{if } x = x_0,\\ x, & \text{if } x \neq x_0. \end{cases}x∧y={x0,y,if x=x0,if x□=x0 ,x∨y={y,x,if x=x0,if x□=x0.

Then (X,∨,∧,x0)(X, \vee, \wedge, x_0)(X,∨,∧,x0) is an Almost Distributive Lattice (ADL), where x0x_0x0 serves as the zero element. This structure is called a discrete ADL. For any a,b∈Xa, b \in Xa,b∈X, we define a partial order "≤\leq≤" by a≤bif and only ifa∧b=a.a \leq b \quad \text{if and only if} \quad a \wedge b = a.a≤bif and only ifa∧b=a.

It is straightforward to verify that "≤\leq≤" is a partial ordering on XXX.

Theorem 2.1 ([3]).

Let (L,∨,∧,0)(L, \vee, \wedge, 0)(L,∨,∧,0) be an ADL with zero element 000. Then, for all a,b,c∈La, b, c \in La,b,c∈L, the following properties hold:

(1) a∧0=0a \wedge 0 = 0a∧0=0 and 0∨a=a0 \vee a = a0∨a=a;
(2) a∧a=a=a∨aa \wedge a = a = a \vee aa∧a=a=a∨a;
(3) (a∧b)∨b=b(a \wedge b) \vee b = b(a∧b)∨b=b, a∨(b∧a)=aa \vee (b \wedge a) = aa∨(b∧a)=a, and a∧(a∨b)=aa \wedge (a \vee b) = aa∧(a∨b)=a;
(4) a∧b=a ⇔ a∨b=ba \wedge b = a \iff a \vee b = ba∧b=a⇔a∨b=b, and a∧b=b ⇔ a∨b=aa \wedge b = b \iff a \vee b = aa∧b=b⇔a∨b=a;
(5) a∧b=b∧aa \wedge b = b \wedge aa∧b=b∧a and a∨b=b∨aa \vee b = b \vee aa∨b=b∨a whenever a≤ba \leq ba≤b;
(6) a∧b≤ba \wedge b \leq ba∧b≤b and a≤a∨ba \leq a \vee ba≤a∨b;
(7) The operation ∧\wedge∧ is associative in LLL;
(8) a∧b∧c=b∧a∧ca \wedge b \wedge c = b \wedge a \wedge ca∧b∧c=b∧a∧c;
(9) (a∨b)∧c=(b∨a)∧c(a \vee b) \wedge c = (b \vee a) \wedge c(a∨b)∧c=(b∨a)∧c;
(10) a∧b=0 ⇔ b∧a=0a \wedge b = 0 \iff b \wedge a = 0a∧b=0⇔b∧a=0;
(11) a∨(b∨a)=a∨ba \vee (b \vee a) = a \vee ba∨(b∨a)=a∨b.

It can be observed that an Almost Distributive Lattice (ADL) LLL satisfies nearly all the properties of a distributive lattice, except possibly the following:

- the right distributivity of ∨\vee∨ over ∧\wedge∧,
- the commutativity of ∨\vee∨,
- the commutativity of ∧\wedge∧, and
- the absorption law, (a∧b)∨a=a(a \wedge b) \vee a = a(a∧b)∨a=a.

If LLL satisfies any one of these properties, then LLL becomes a distributive lattice.

Theorem 2.2 ([3]).

Let (L,∨,∧,0)(L, \vee, \wedge, 0)(L,∨,∧,0) be an ADL with zero element 000. Then the following statements are equivalent:

1. (L,∨,∧,0)(L, \vee, \wedge, 0)(L,∨,∧,0) is a distributive lattice;
2. a∨b=b∨aa \vee b = b \vee aa∨b=b∨a, for all a,b∈La, b \in La,b∈L;
3. a∧b=b∧aa \wedge b = b \wedge aa∧b=b∧a, for all a,b∈La, b \in La,b∈L;
4. (a∧b)∨c=(a∨c)∧(b∨c)(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)(a∧b)∨c=(a∨c)∧(b∨c), for all a,b,c∈La, b, c \in La,b,c∈L.

Proposition 2.1 ([3]).

Let (L,∨,∧)(L, \vee, \wedge)(L,∨,∧) be an Almost Distributive Lattice (ADL). Then, for any a,b,c∈La, b, c \in La,b,c∈L with a≤ba \leq ba≤b, the following hold:

1. a∧c≤b∧ca \wedge c \leq b \wedge ca∧c≤b∧c;
2. c∧a≤c∧bc \wedge a \leq c \wedge bc∧a≤c∧b;
3. c∨a≤c∨bc \vee a \leq c \vee bc∨a≤c∨b.

Definition 2.2 ([3]).

An element m∈Lm \in Lm∈L is said to be maximal if it is maximal in the partially ordered set (L,≤)(L, \leq)(L,≤); that is, for any a∈La \in La∈L, m≤a ⟹ m=a.m \leq a \implies m = a.m≤a⟹m=a.

Theorem 2.3 ([3]).

Let LLL be an ADL and m∈Lm \in Lm∈L. Then the following statements are equivalent:

1. mmm is maximal with respect to ≤\leq≤;
2. m∨a=mm \vee a = mm∨a=m, for all a∈La \in La∈L;
3. m∧a=am \wedge a = am∧a=a, for all a∈La \in La∈L.
4. Lemma 2.1 ([3]).

Let LLL be an Almost Distributive Lattice (ADL) with a maximal element mmm, and let x,y∈Lx, y \in Lx,y∈L.
If x∧y=yx \wedge y = yx∧y=y and y∧x=xy \wedge x = xy∧x=x, then xxx is maximal if and only if yyy is maximal.

Moreover, the following conditions are equivalent:

(i) x∧y=yx \wedge y = yx∧y=y and y∧x=xy \wedge x = xy∧x=x;
(ii) x∧m=y∧mx \wedge m = y \wedge mx∧m=y∧m.

Definition 2.3 ([3]).

If (L,∨,∧,0,m)(L, \vee, \wedge, 0, m)(L,∨,∧,0,m) is an ADL with zero element 000 and maximal element mmm, then the set I(L)I(L)I(L) of all ideals of LLL forms a complete lattice under set inclusion.

In this lattice, for any I,J∈I(L)I, J \in I(L)I,J∈I(L), the least upper bound (l.u.b.) and greatest lower bound (g.l.b.) are given by: I∨J={(x∨y)∧m∣x∈I, y∈J},I∧J=I∩J.I \vee J = \{ (x \vee y) \wedge m \mid x \in I,\, y \in J \}, \quad I \wedge J = I \cap J.I∨J={(x∨y)∧m∣x∈I,y∈J},I∧J=I∩J.

Furthermore, the set PI(L)={(a]∣a∈L},PI(L) = \{ (a] \mid a \in L \},PI(L)={(a]∣a∈L},

consisting of all principal ideals of LLL, forms a sublattice of I(L)I(L)I(L). Since (a]∨(b]=(a∨b](a] \vee (b] = (a \vee b](a]∨(b]=(a∨b] and (a]∩(b]=(a∧b](a] \cap (b] = (a \wedge b](a]∩(b]=(a∧b], we now introduce the concepts of skew semi-residuation and multiplication in an Almost Distributive Lattice (ADL) LLL, along with the definition of a Skew Semi-Residuated Almost Distributive Lattice (SSR-ADL), as presented in our earlier paper [2].

Definition 2.4 ([2]).

Let LLL be an ADL with a maximal element mmm. A binary operation "∘\circ∘" on LLL is called a skew semi-residuation over LLL if, for all a,b,c∈La, b, c \in La,b,c∈L, the following conditions are satisfied: (R1) a∧b=ba \wedge b = ba∧b=b if and only if a∘ba \circ ba∘b is maximal.

(R2) If a∧b=ba \wedge b = ba∧b=b, then:
(i) (a∘c)∧(b∘c)=b∘c(a \circ c) \wedge (b \circ c) = b \circ c(a∘c)∧(b∘c)=b∘c;
(ii) (c∘b)∧(c∘a)=c∘a(c \circ b) \wedge (c \circ a) = c \circ a(c∘b)∧(c∘a)=c∘a.

(R3) [(a∧b)∘c]∧m=(a∘c)∧(b∘c)∧m.[(a \wedge b) \circ c] \wedge m = (a \circ c) \wedge (b \circ c) \wedge m.[(a∧b)∘c]∧m=(a∘c)∧(b∘c)∧m.

(R4) [c∘(a∨b)]∧m=(c∘a)∧(c∘b)∧m.[c \circ (a \vee b)] \wedge m = (c \circ a) \wedge (c \circ b) \wedge m.[c∘(a∨b)]∧m=(c∘a)∧(c∘b)∧m.

Definition 2.5 ([4]).

Let LLL be an ADL with a maximal element mmm. A binary operation "·\cdot·" on LLL is called a multiplication over LLL if, for all a,b,c∈La, b, c \in La,b,c∈L, the following conditions are satisfied: *(conditions would follow here in your text)*

(M1) (a·b)∧m=(b·a)∧m(a \cdot b) \wedge m = (b \cdot a) \wedge m(a·b)∧m=(b·a)∧m
(M2) [(a·b)·c]∧m=[a·(b·c)]∧m[(a \cdot b) \cdot c] \wedge m = [a \cdot (b \cdot c)] \wedge m[(a·b)·c]∧m=[a·(b·c)]∧m

(M3) (a·m)∧m=a∧m(a \cdot m) \wedge m = a \wedge m(a·m)∧m=a∧m
(M4) [a·(b∨c)]∧m=[(a·b)∨(a·c)]∧m[a \cdot (b \vee c)] \wedge m = [(a \cdot b) \vee (a \cdot c)] \wedge m[a·(b∨c)]∧m=[(a·b)∨(a·c)]∧m

Definition 2.6 ([2]).

An Almost Distributive Lattice (ADL) LLL with a maximal element mmm is said to be a Skew Semi-Residuated Almost Distributive Lattice (SSR-ADL) if there exist two binary operations, "∘\circ∘" and "·\cdot·", on LLL satisfying conditions (R1)–(R4), (M1)–(M4), and the following condition: (A) (x∘a)∧b=b ⟺ x∧(a·b)=a·b,(x \circ a) \wedge b = b \iff x \wedge (a \cdot b) = a \cdot b,(x∘a)∧b=b⟺x∧(a·b)=a·b, for all x,a,b∈L.x, a, b \in L.x,a,b∈L.

The following properties are used frequently in subsequent results.

Lemma 2.2 ([2]).

Let LLL be an **ADL** with a maximal element mmm, and let "·\cdot·" be a binary operation on LLL satisfying conditions (M1)–(M4). Then, for any a,b,c,d∈La, b, c, d \in La,b,c,d∈L, the following hold:

(i) a∧(a·b)=a·ba \wedge (a \cdot b) = a \cdot ba∧(a·b)=a·b and b∧(a·b)=a·bb \wedge (a \cdot b) = a \cdot bb∧(a·b)=a·b;

(ii) (ii) If a∧b=ba \wedge b = ba∧b=b, then (c·a)∧(c·b)=c·b(c \cdot a) \wedge (c \cdot b) = c \cdot b(c·a)∧(c·b)=c·b and (a·c)∧(b·c)=b·c(a \cdot c) \wedge (b \cdot c) = b \cdot c(a·c)∧(b·c)=b·c;

(iii) d∧[(a·b)·c]=(a·b)·c ⟺ d∧[a·(b·c)]=a·(b·c)d \wedge [(a \cdot b) \cdot c] = (a \cdot b) \cdot c \iff d \wedge [a \cdot (b \cdot c)] = a \cdot (b \cdot c)d∧[(a·b)·c]=(a·b)·c⟺d∧[a·(b·c)]=a·(b·c);

(iv) (a·c)∧(b·c)∧[(a∧b)·c]=(a∧b)·c(a \cdot c) \wedge (b \cdot c) \wedge [(a \wedge b) \cdot c] = (a \wedge b) \cdot c(a·c)∧(b·c)∧[(a∧b)·c]=(a∧b)·c; (v) If d∧(a·c)∧(b·c)=(a·c)∧(b·c)d \wedge (a \cdot c) \wedge (b \cdot c) = (a \cdot c) \wedge (b \cdot c)d∧(a·c)∧(b·c)=(a·c)∧(b·c), then d∧[(a∧b)·c]=(a∧b)·cd \wedge [(a \wedge b) \cdot c] = (a \wedge b) \cdot cd∧[(a∧b)·c]=(a∧b)·c;

(v) d∧[(a·c)∨(b·c)]=(a·c)∨(b·c) ⟺ d∧[(a∨b)·c]=(a∨b)·c.d \wedge [(a \cdot c) \vee (b \cdot c)] = (a \cdot c) \vee (b \cdot c) \iff d \wedge [(a \vee b) \cdot c] = (a \vee b) \cdot c.d∧[(a·c)∨(b·c)]=(a·c)∨(b·c)⟺d∧[(a∨b)·c]=(a∨b)·c.

The following result is a direct consequence of condition (M1) in Definition 2.5.

Lemma 2.3 ([2]).

Let LLL be an ADL with a maximal element mmm, and let “·\cdot·” be a binary operation on LLL satisfying condition (M1). For all a,b,x∈La, b, x \in La,b,x∈L, a∧(x·b)=x·b ⟺ a∧(b·x)=b·x.a \wedge (x \cdot b) = x \cdot b \iff a \wedge (b \cdot x) = b \cdot x.a∧(x·b)=x·b⟺a∧(b·x)=b·x.

In the following, we present several important properties of skew semi-residuation “∘\circ∘” and multiplication “·\cdot·” in a skew semi-residuated ADL LLL, as derived from our earlier work [2].

Lemma 2.4 ([2]).

Let LLL be a skew semi-residuated ADL with a maximal element mmm. For all a,b,c,d∈La, b, c, d \in La,b,c,d∈L, the following properties hold:

(1) (a∘b)∧a=a(a \circ b) \wedge a = a(a∘b)∧a=a
(2) [a∘(a∘b)]∧(a∨b)=a∨b[a \circ (a \circ b)] \wedge (a \vee b) = a \vee b[a∘(a∘b)]∧(a∨b)=a∨b
(3) [(a∘b)∘c]∧[a∘(b·c)]=a∘(b·c)[(a \circ b) \circ c] \wedge [a \circ (b \cdot c)] = a \circ (b \cdot c)[(a∘b)∘c]∧[a∘(b·c)]=a∘(b·c)
(4) [a∘(b·c)]∧[(a∘b)∘c]=(a∘b)∘c[a \circ (b \cdot c)] \wedge [(a \circ b) \circ c] = (a \circ b) \circ c[a∘(b·c)]∧[(a∘b)∘c]=(a∘b)∘c
(5) [(a∧b)∘b]∧(a∘b)=a∘b[(a \wedge b) \circ b] \wedge (a \circ b) = a \circ b[(a∧b)∘b]∧(a∘b)=a∘b
(6) (a∘b)∧[(a∧b)∘b]=(a∧b)∘b(a \circ b) \wedge [(a \wedge b) \circ b] = (a \wedge b) \circ b(a∘b)∧[(a∧b)∘b]=(a∧b)∘b
(7) [a∘(a∨b)]∧m=(a∘b)∧m[a \circ (a \vee b)] \wedge m = (a \circ b) \wedge m[a∘(a∨b)]∧m=(a∘b)∧m
(8) [c∘(a∧b)]∧[(c∘a)∨(c∘b)]=(c∘a)∨(c∘b)[c \circ (a \wedge b)] \wedge [(c \circ a) \vee (c \circ b)] = (c \circ a) \vee (c \circ b)[c∘(a∧b)]∧[(c∘a)∨(c∘b)]=(c∘a)∨(c∘b)
(9) If a∘b=aa \circ b = aa∘b=a and a∧(b·d)=b·da \wedge (b \cdot d) = b \cdot da∧(b·d)=b·d, then a∧d=da \wedge d = da∧d=d

(10) {a∘[a∘(a∘b)]}∧(a∘b)=a∘b\{a \circ [a \circ (a \circ b)]\} \wedge (a \circ b) = a \circ b{a∘[a∘(a∘b)]}∧(a∘b)=a∘b

(11) [(a∨b)∘c]∧[(a∘c)∨(b∘c)]=(a∘c)∨(b∘c)[(a \vee b) \circ c] \wedge [(a \circ c) \vee (b \circ c)] = (a \circ c) \vee (b \circ c)[(a∨b)∘c]∧[(a∘c)∨(b∘c)]=(a∘c)∨(b∘c)

(12) If a∧m≥b∧ma \wedge m \ge b \wedge ma∧m≥b∧m, then (a∘c)∧m≥(b∘c)∧m(a \circ c) \wedge m \ge (b \circ c) \wedge m(a∘c)∧m≥(b∘c)∧m

(13) (a∘b)∧{a∘[a∘(a∘b)]}=a∘[a∘(a∘b)](a \circ b) \wedge \{a \circ [a \circ (a \circ b)]\} = a \circ [a \circ (a \circ b)](a∘b)∧{a∘[a∘(a∘b)]}=a∘[a∘(a∘b)]

(14) If a∧b=ba \wedge b = ba∧b=b, then (a·c)∧(b·c)=b·c(a \cdot c) \wedge (b \cdot c) = b \cdot c(a·c)∧(b·c)=b·c

(15) a∧b∧(a·b)=a·ba \wedge b \wedge (a \cdot b) = a \cdot ba∧b∧(a·b)=a·b

(16) [(a·b)∘a]∧b=b[(a \cdot b) \circ a] \wedge b = b[(a·b)∘a]∧b=b

(17) (a·b)∧[(a∧b)·(a∨b)]=(a∧b)·(a∨b)(a \cdot b) \wedge [(a \wedge b) \cdot (a \vee b)] = (a \wedge b) \cdot (a \vee b)(a·b)∧[(a∧b)·(a∨b)]=(a∧b)·(a∨b)

(18) If a∨ba \vee ba∨b is maximal, then (a·b)∧a∧b=a∧b(a \cdot b) \wedge a \wedge b = a \wedge b(a·b)∧a∧b=a∧b

Noether Almost Distributive Lattices

In this section, we introduce the concepts of a Noether Skew Semi Residuated Almost Distributive Lattice (Noether SSR ADL) and a principal element in a Skew Semi Residuated Almost Distributive Lattice (SSR ADL). We establish important results within a Noether SSR ADL and also prove the fundamental theorem for Noether SSR ADLs. We begin by defining several key concepts in a skew semi-residuated ADL **LLL:**

Definition 3.1.

An element c∈Lc \in Lc∈L is called irreducible if, for any f,g∈Lf, g \in Lf,g∈L,
f∧g=c ⟹ f=c or g=c.f \wedge g = c \implies f = c \text{ or } g = c.f∧g=c⟹f=c or g=c.

Definition 3.2.

An element p∈Lp \in Lp∈L is called prime if, for any a,b∈La, b \in La,b∈L,

p∧(a·b)=a·b ⟹ p∧a=a or p∧b=b.p \wedge (a \cdot b) = a \cdot b \implies p \wedge a = a \text{ or } p \wedge b = b.p∧(a·b)=a·b⟹p∧a=a or p∧b=b.

Definition 3.3.

An element p∈Lp \in Lp∈L is called primary if, for any a,b∈La, b \in La,b∈L, p∧(a·b)=a·b and p∧a≠a ⟹ p∧bs=bs for some s∈Z+.p \wedge (a \cdot b) = a \cdot b \text{ and } p \wedge a \neq a \implies p \wedge b^s = b^s \text{ for some } s \in \mathbb{Z}^+.p∧(a·b)=a·b and p∧a□=a⟹p∧bs=bs for some s∈Z+.

Definition 3.4.

An ADL LLL is said to satisfy the ascending chain condition (a.c.c.) if, for every increasing sequence x1≤x2≤x3≤… in L,x_1 \leq x_2 \leq x_3 \leq \dots \text{ in } L,x1 ≤x2≤x3≤… in L, there exists a positive integer nnn such that xn=xn+1=xn+2=…x_n = x_{n+1} = x_{n+2} = \dotsxn=xn+1=xn+2=…

Definition 3.5.

A skew semi-residuated ADL LLL is called a Noether Skew Semi Residuated ADL (Noether SSR ADL) if it satisfies the following conditions:

1. (N1) The ascending chain condition (a.c.c.) holds in LLL.
2. (N2) Every irreducible element of LLL is primary.

Definition 3.6.

An element a∈La \in La∈L is said to have a primary decomposition if there exist primary elements p1,p2,…,pm∈Lp_1, p_2, \dots, p_m \in Lp1,p2,…,pm∈L such that a=p1∧p2∧⋯∧pm.a = p_1 \wedge p_2 \wedge \dots \wedge p_m.a=p1∧p2∧⋯∧pm

Theorem 3.1.

Let a,b∈La, b \in La,b∈L in a Noether SSR ADL LLL, and suppose a·ba \cdot ba·b has a primary decomposition. Then there exists an exponent sss such that (a·b)∧a∧bs=a∧bs.(a \cdot b) \wedge a \wedge b^s = a \wedge b^s.(a·b)∧a∧bs=a∧bs.

Proof

Let LLL be a Noether SSR ADL and a,b∈La, b \in La,b∈L. Assume that

a·b=p1∧p2∧⋯∧pka \cdot b = p_1 \wedge p_2 \wedge \dots \wedge p_ka·b=p1∧p2∧⋯∧pk is a primary decomposition of a·ba \cdot ba·b. Then, for each i=1,2,…,ki = 1, 2, \dots, ki=1,2,…,k, we have pi∧(a·b)=a·b.p_i \wedge (a \cdot b) = a \cdot b.pi∧(a·b)=a·b.

By the definition of primary elements, for each iii, either pi∧a=ap_i \wedge a = api∧a=a or pi∧a≠ap_i \wedge a \neq api∧a□=a and there exists a positive integer sis_isi such that pi∧bsi=bsi.p_i \wedge b^{s_i} = b^{s_i}.pi∧bsi=bsi.

Taking s=max{s1,s2,…,sk}s = \max\{s_1, s_2, \dots, s_k\}s=max{s1,s2,…,sk}, it follows that (a·b)∧a∧bs=a∧bs.(a \cdot b) \wedge a \wedge b^s = a \wedge b^s.(a·b)∧a∧bs=a∧bs.

Case (i): Suppose pk∧a=ap_k \wedge a = apk∧a=a. Rearrange the primary elements p1,p2,…,pk−1p_1, p_2, \dots, p_{k-1}p1,p2,…,pk−1 such that pi∧bsi=bsi,1≤i≤l,p_i \wedge b_{s_i} = b_{s_i}, \quad 1 \le i \le l, pi∧bsi=bsi,1≤i≤l, and pi∧a=a,l+1≤i≤k.p_i \wedge a = a, \quad l+1 \le i \le k.pi∧a=a,l+1≤i≤k. Let s=max{s1,s2,…,sl}s = \max\{s_1, s_2, \dots, s_l\}s=max{s1,s2,…,sl}. Then pi∧bs=bs,1≤i≤l.p_i \wedge b_s = b_s, \quad 1 \le i \le l.pi∧bs=bs,1≤i≤l.

Now, consider (a·b)∧a∧bs=p1∧p2∧⋯∧pk∧a∧bs.(a \cdot b) \wedge a \wedge b_s = p_1 \wedge p_2 \wedge \dots \wedge p_k \wedge a \wedge b_s.(a·b)∧a∧bs=p1∧p2∧⋯∧pk∧a∧bs.

Rewriting by grouping, we get pl+1∧⋯∧pk∧a∧p1∧⋯∧pl∧bs=a∧bs.p_{l+1} \wedge \dots \wedge p_k \wedge a \wedge p_1 \wedge \dots \wedge p_l \wedge b_s = a \wedge b_s.pl+1∧⋯∧pk∧a∧p1∧⋯∧pl∧bs=a∧bs.

Case (ii): Suppose pk∧a≠ap_k \wedge a \ne apk∧a=a and pk∧bsk=bskp_k \wedge b_{s_k} = b_{s_k}pk∧bsk=bsk. Rearrange the primary elements p1,p2,…,pk−1p_1, p_2, \dots, p_{k-1}p1,p2,…,pk−1 such that pi∧a=a,1≤i≤j,p_i \wedge a = a, \quad 1 \le i \le j,pi∧a=a,1≤i≤j, and pi∧bsi=bsi,j+1≤i≤k−1.p_i \wedge b_{s_i} = b_{s_i}, \quad j+1 \le i \le k-1.pi∧bsi=bsi,j+1≤i≤k−1.

Let s=max{sj+1,sj+2,…,sk}s = \max\{s_{j+1}, s_{j+2}, \dots, s_k\}s=max{sj+1,sj+2,…,sk}. Then pi∧bs=bs,j+1≤i≤k.p_i \wedge b_s = b_s, \quad j+1 \le i \le k.pi∧bs=bs,j+1≤i≤k. Now, (a·b)∧a∧bs=p1∧⋯∧pk∧a∧bs.(a \cdot b) \wedge a \wedge b_s = p_1 \wedge \dots \wedge p_k \wedge a \wedge b_s.(a·b)∧a∧bs=p1∧⋯∧pk∧a∧bs.

Grouping appropriately, we get p1∧⋯∧pj∧a∧pj+1∧⋯∧pk∧bs=a∧bs.p_1 \wedge \dots \wedge p_j \wedge a \wedge p_{j+1} \wedge \dots \wedge p_k \wedge b_s = a \wedge b_s.p1∧⋯∧pj∧a∧pj+1∧⋯∧pk∧bs=a∧bs.

Theorem 3.2. Let LLL be a Noetherian SSR ADL with a maximal element mmm, and let a,b,c∈La, b, c \in La,b,c∈L. If b·b=bb \cdot b = bb·b=b, then

1. a∧b∧m=(a·b)∧ma \wedge b \wedge m = (a \cdot b) \wedge ma∧b∧m=(a·b)∧m, and
2. [(a∧c)·b]∧m=(a·b)∧(c·b)∧m[(a \wedge c) \cdot b] \wedge m = (a \cdot b) \wedge (c \cdot b) \wedge m[(a∧c)·b]∧m=(a·b)∧(c·b)∧m.

PROOF

Let a,b,c∈La, b, c \in La,b,c∈L and suppose b·b=bb \cdot b = bb·b=b.

(i) By property (15) of Lemma 2.4,

a∧b∧(a·b)=a·b.a \wedge b \wedge (a \cdot b) = a \cdot b.a∧b∧(a·b)=a·b. Also, by **Theorem 3.1,**
(a·b)∧a∧bs=a∧bs.(a \cdot b) \wedge a \wedge b_s = a \wedge b_s.(a·b)∧a∧bs=a∧bs. Since b·b=bb \cdot b = bb·b=b, we have (a·b)∧a∧b=a∧b.(a \cdot b) \wedge a \wedge b = a \wedge b.(a·b)∧a∧b=a∧b.
Hence,

a∧b∧m=(a·b)∧m.a \wedge b \wedge m = (a \cdot b) \wedge m.a∧b∧m=(a·b)∧m.

(ii) Using (i),

[(a∧c)·b]∧m=a∧c∧b∧m.[(a \wedge c) \cdot b] \wedge m = a \wedge c \wedge b \wedge m.[(a∧c)·b]∧m=a∧c∧b∧m.

Rewriting, a∧c∧b∧m=a∧b∧m∧c∧b∧m.a \wedge c \wedge b \wedge m = a \wedge b \wedge m \wedge c \wedge b \wedge m.a∧c∧b∧m=a∧b∧m∧c∧b∧m.

By (i) again a∧b∧m∧c∧b∧m=(a·b)∧m∧(c·b)∧m.a \wedge b \wedge m \wedge c \wedge b \wedge m = (a \cdot b) \wedge m \wedge (c \cdot b) \wedge m.a∧b∧m∧c∧b∧m=(a·b)∧m∧(c·b)∧m.

Therefore, [(a∧c)·b]∧m=(a·b)∧(c·b)∧m.[(a \wedge c) \cdot b] \wedge m = (a \cdot b) \wedge (c \cdot b) \wedge m.[(a∧c)·b]∧m=(a·b)∧(c·b)∧m.

COROLLARY 3.1.

Let LLL be a Noetherian SSR ADL with maximal element mmm. If aaa and bbb are idempotent elements of LLL, then (a·b)∧m=a∧b∧m.(a \cdot b) \wedge m = a \wedge b \wedge m.(a·b)∧m=a∧b∧m.

PROOF

Let a,b∈La, b \in La,b∈L be idempotent, i.e., a2=aa^2 = aa2=a and b2=bb^2 = bb2=b.

By property (15) of Lemma 2.4, a∧b∧(a·b)=a·b.a \wedge b \wedge (a \cdot b) = a \cdot b.a∧b∧(a·b)=a·b.

By Theorem 3.1, (a·b)∧a∧bs=a∧bs.(a \cdot b) \wedge a \wedge b_s = a \wedge b_s.(a·b)∧a∧bs=a∧bs.

Hence,

(a·b)∧a∧b=a∧b.(a \cdot b) \wedge a \wedge b = a \wedge b.(a·b)∧a∧b=a∧b.

Therefore, (a·b)∧m=a∧b∧m.(a \cdot b) \wedge m = a \wedge b \wedge m.(a·b)∧m=a∧b∧m..

The following theorem is the converse of Theorem 3.1 under special conditions.

THEOREM 3.3

Let LLL be a skew semi-residuated ADL with the ascending chain condition (a.c.c.). Suppose that for any a,b∈La, b \in La,b∈L, there exists s∈Z+s \in \mathbb{Z}^+s∈Z+ such that (a·b)∧a∧bs=a∧bs.(a \cdot b) \wedge a \wedge b_s = a \wedge b_s.(a·b)∧a∧bs =a∧bs.

Then LLL is a Noetherian SSR ADL.

PROOF

Let ppp be an irreducible element of LLL. Let a,b∈La, b \in La,b∈L satisfy p∧(a·b)=a·bandp∧a≠a.p \wedge (a \cdot b) = a \cdot b \quad \text{and} \quad p \wedge a \ne a.p∧(a·b)=a·bandp∧a□=a.

Choose s∈Z+s \in \mathbb{Z}^+s∈Z+ such that (a·b)∧a∧bs=a∧bs.(a \cdot b) \wedge a \wedge b_s = a \wedge b_s.(a·b)∧a∧bs=a∧bs.

Then,

p∧(a·b)=a·b ⟹ p∧(a·b)∧a∧bs=(a·b)∧a∧bs.p \wedge (a \cdot b) = a \cdot b \implies p \wedge (a \cdot b) \wedge a \wedge b_s = (a \cdot b) \wedge a \wedge b_s.p∧(a·b)=a·b⟹p∧(a·b)∧a∧bs=(a·b)∧a∧bs.

Hence,

p∧a∧bs=a∧bs ⟹ p∨(a∧bs)=p.p \wedge a \wedge b_s = a \wedge b_s \implies p \vee (a \wedge b_s) = p.p∧a∧bs=a∧bs⟹p∨(a∧bs)=p.

Using the distributive property,

(p∨a)∧(p∨bs)=p.(p \vee a) \wedge (p \vee b_s) = p.(p∨a)∧(p∨bs)=p.

Since ppp is irreducible, this implies either

p∨a=porp∨bs=p.p \vee a = p \quad \text{or} \quad p \vee b_s = p.p∨a=porp∨bs=p. But p∨a≠pp \vee a \ne pp∨a□=p because p∧a≠ap \wedge a \ne ap∧a□=a. Therefore, p∨bs=p.p \vee b_s = p.p∨bs=p.

This completes the proof.

Thus, p∧bs=bsfor some s∈Z+.p \wedge b_s = b_s \quad \text{for some } s \in \mathbb{Z}^+.p∧bs=bsfor some s∈Z+.

Therefore, ppp is a primary element.

Hence, every irreducible element of LLL is primary. Thus, LLL is a Noetherian SSR ADL.

DEFINITION 3.7.

Let LLL be a skew semi-residuated ADL. An element a∈La \in La∈L is called **principal** if, whenever a∧b=ba \wedge b = ba∧b=b for some b∈Lb \in Lb∈L, there exists c∈Lc \in Lc∈L such that a·c=b.a \cdot c = b.a·c=b.

LEMMA 3.1.

Let LLL be a skew semi-residuated ADL with a maximal element mmm. If a,b∈La, b \in La,b∈L such that aaa is principal and a∧b=ba \wedge b = ba∧b=b, then

[(b∘a)·a]∧m=b∧m.[(b \circ a) \cdot a] \wedge m = b \wedge m.[(b∘a)·a]∧m=b∧m.

PROOF

Let a,b∈La, b \in La,b∈L with aaa principal and a∧b=ba \wedge b = ba∧b=b. Since aaa is principal, there exists c∈Lc \in Lc∈L such that a·c=b.a \cdot c = b.a·c=b.

Then, b∧(a·c)=a·c ⟹ (b∘a)∧c=c(by Definition 2.6).b \wedge (a \cdot c) = a \cdot c \implies (b \circ a) \wedge c = c \quad \text{(by Definition 2.6)}.b∧(a·c)=a·c⟹(b∘a)∧c=c(by Definition 2.6).

By Lemma 2.2(ii), [(b∘a)·a]∧(c·a)=c·a.[(b \circ a) \cdot a] \wedge (c \cdot a) = c \cdot a.[(b∘a)·a]∧(c·a)=c·a.

By Lemma 2.3, [(b∘a)·a]∧(a·c)=a·c.[(b \circ a) \cdot a] \wedge (a \cdot c) = a \cdot c.[(b∘a)·a]∧(a·c)=a·c.

Since a·c=ba \cdot c = ba·c=b, it follows that

[(b∘a)·a]∧b=b.[(b \circ a) \cdot a] \wedge b = b.[(b∘a)·a]∧b=b.

Also, (b∘a)∧(b∘a)=b∘a ⟹ b∧[a·(b∘a)]=a·(b∘a)(by Definition 2.6).(b \circ a) \wedge (b \circ a) = b \circ a \implies b \wedge [a \cdot (b \circ a)] = a \cdot (b \circ a) \quad \text{(by Definition 2.6)}.(b∘a)∧(b∘a)=b∘a⟹b∧[a·(b∘a)]=a·(b∘a)(by Definition 2.6).

By Lemma 2.3,

b∧[(b∘a)·a]=(b∘a)·a.b \wedge [(b \circ a) \cdot a] = (b \circ a) \cdot a.b∧[(b∘a)·a]=(b∘a)·a.

Hence,

[(b∘a)·a]∧m=b∧m.[(b \circ a) \cdot a] \wedge m = b \wedge m.[(b∘a)·a]∧m=b∧m.

We now prove the following fundamental theorem:

THEOREM 3.4.

Let LLL be a skew semi-residuated ADL with a maximal element mmm, satisfying the following conditions:

1. LLL satisfies the ascending chain condition (a.c.c.).
2. Every element of LLL is principal.

Then LLL is a Noetherian Skew Semi-Residuated ADL.

PROOF.

Assume that conditions (1) and (2) hold in LLL. Suppose, for the sake of contradiction, that p∈Lp \in Lp∈L is a non-primary element.

Then there exist a,b∈La, b \in La,b∈L such that p∧(a·b)=a·b,p∧a≠a,andp∧bs≠bsfor all s∈Z+.p \wedge (a \cdot b) = a \cdot b, \quad p \wedge a \ne a, \quad \text{and} \quad p \wedge b_s \ne b_s \quad \text{for all } s \in \mathbb{Z}^+.p∧(a·b)=a·b,p∧a□=a,andp∧bs□=bsfor all s∈Z+.

Let k∈Z+k \in \mathbb{Z}^+k∈Z+. By Lemma 2.2(i), bk−1∧bk=bk.b_{k-1} \wedge b_k = b_k.bk−1∧bk=bk.
Hence,
(p∘bk)∧(p∘bk−1)=p∘bk−1 ⟹ (p∘bk)∧m≥(p∘bk−1)∧m.(p \circ b_k) \wedge (p \circ b_{k-1}) = p \circ b_{k-1} \implies (p \circ b_k) \wedge m \ge (p \circ b_{k-1}) \wedge m.(p∘bk)∧(p∘bk−1)=p∘bk−1⟹(p∘bk)∧m≥(p∘bk−1)∧m. Since LLL satisfies the ascending chain condition (a.c.c.), the chain (p∘b)∧m≤(p∘b2)∧m≤(p∘b3)∧m≤…(p \circ b) \wedge m \le (p \circ b_2) \wedge m \le (p \circ b_3) \wedge m \le \dots(p∘b)∧m≤(p∘b2)∧m≤(p∘b3)∧m≤… must terminate.
Then, there exists k∈Z+k \in \mathbb{Z}^+k∈Z+ such that (p∘bk)∧m=(p∘bk+1)∧m(i)(p \circ b_k) \wedge m = (p \circ b_{k+1}) \wedge m \tag{i}(p∘bk)∧m=(p∘bk+1)∧m(i)
Define c=(p∨a)∧(p∨bk).c = (p \vee a) \wedge (p \vee b_k).c=(p∨a)∧(p∨bk).

Then

p≤c≤p∨bk.p \le c \le p \vee b_k.p≤c≤p∨bk.

Now,

c=c∧(p∨bk)=p∨(c∧bk)(ii)c = c \wedge (p \vee b_k) = p \vee (c \wedge b_k) \tag{ii}c=c∧(p∨bk)=p∨(c∧bk)(ii)

Step 1: First, we show that

p∧c∧bk=c∧bk.p \wedge c \wedge b_k = c \wedge b_k.p∧c∧bk=c∧bk.

Since bkb_kbk is principal and

bk∧c∧bk=c∧bk,b_k \wedge c \wedge b_k = c \wedge b_k,bk∧c∧bk=c∧bk,

it follows that

c∧bk∧m=([(c∧bk)∘bk]·bk)∧m(by Lemma 3.1)(iii)c \wedge b_k \wedge m = ([(c \wedge b_k) \circ b_k] \cdot b_k) \wedge m \tag{iii} \quad (\text{by Lemma 3.1})c∧bk ∧m=([(c∧bk)∘bk]·bk)∧m(by Lemma 3.1)(iii)

Step 2: Since (p∨a)∧c=c(p \vee a) \wedge c = c(p∨a)∧c=c, by Lemma 2.2(ii), we have

[b·(p∨a)]∧(b·c)=b·c.[b \cdot (p \vee a)] \wedge (b \cdot c) = b \cdot c.[b·(p∨a)]∧(b·c)=b·c.

Then,

p∧(b·c)=p∧[b·(p∨a)]∧(b·c)=p∧[(b·p)∨(b·a)]∧(b·c)=[(b·p)∨(b·a)]∧(b·c)=[b·(p∨a)]∧(b·c)= b·c(since (p∨a)∧c=c).\begin{aligned} p \wedge (b \cdot c) &= p \wedge [b \cdot (p \vee a)] \wedge (b \cdot c) \\ &= p \wedge [(b \cdot p) \vee (b \cdot a)] \wedge (b \cdot c) \\ &= [(b \cdot p) \vee (b \cdot a)] \wedge (b \cdot c) \\ &= [b \cdot (p \vee a)] \wedge (b \cdot c) \\ &= b \cdot c \quad (\text{since } (p \vee a) \wedge c = c). \end{aligned}p∧(b·c) =p∧[b·(p∨a)]∧(b·c)=p∧[(b·p)∨(b·a)]∧(b·c)=[(b·p)∨(b·a)]∧(b·c)=[b·(p∨a)]∧(b·c)=b·c(since (p∨a)∧c=c).

Step 3: Consequently,

p∧(b·c)∧[b·(c∧bk)]=(b·c)∧[b·(c∧bk)] ⟹ p∧[b·(c∧bk)]=b·(c∧bk)p \wedge (b \cdot c) \wedge [b \cdot (c \wedge b_k)] = (b \cdot c) \wedge [b \cdot (c \wedge b_k)] \implies p \wedge [b \cdot (c \wedge b_k)] = b \cdot (c \wedge b_k)p∧(b·c)∧[b·(c∧bk)]=(b·c)∧[b·(c∧bk)]⟹p∧[b·(c∧bk)]=b·(c∧bk)

(by the idempotence of c∧bkc \wedge b_kc∧bk).

Hence, by Definition 2.6,

(p∘b)∧c∧bk=c∧bk ⟹ (p∘b)∧c∧bk∧m=c∧bk∧m.(p \circ b) \wedge c \wedge b_k = c \wedge b_k \implies (p \circ b) \wedge c \wedge b_k \wedge m = c \wedge b_k \wedge m.(p∘b)∧c∧bk=c∧bk⟹(p∘b)∧c∧bk∧m=c∧bk∧m.

Using (iii),

(p∘b)∧([(c∧bk)∘bk]·bk)∧m=([(c∧bk)∘bk]·bk)∧m,(p \circ b) \wedge ([(c \wedge b_k) \circ b_k] \cdot b_k) \wedge m = ([(c \wedge b_k) \circ b_k] \cdot b_k) \wedge m,(p∘b)∧([(c∧bk)∘bk]·bk)∧m=([(c∧bk)∘bk]·bk)∧m,

and hence,

p∧(c∧bk)∧m=c∧bk∧m ⟹ p∧c∧bk=c∧bk.p \wedge (c \wedge b_k) \wedge m = c \wedge b_k \wedge m \implies p \wedge c \wedge b_k = c \wedge b_k.p∧(c∧bk)∧m=c∧bk∧m⟹p∧c∧bk=c∧bk.

Therefore,

p∨(c∧bk)=p ⟹ c=p∨(c∧bk)=p.p \vee (c \wedge b_k) = p \implies c = p \vee (c \wedge b_k) = p.p∨(c∧bk)=p⟹c=p∨(c∧bk)=p.

But

p=(p∨a)∧(p∨bk),p = (p \vee a) \wedge (p \vee b_k),p=(p∨a)∧(p∨bk),

and since p∨a≠pp \vee a \ne pp∨a=p and p∨bk≠pp \vee b_k \ne pp∨bk=p, it follows that ppp is **reducible**.

L is a Noetherian Skew Semi-Residuated ADL.

References

1. Dilworth, R.P., *Abstract residuation over lattices*, Bulletin of the American Mathematical Society, Vol. 44 (1938), pp. 262–268.

2. Pavulurimachara Srinivas, Madhusudan Patro, and S.S. Raju, *Skew Semi-Residuated Almost Distributive Lattices – I*, Nanotechnology Perceptions, Vol. 20, No. 6 (2024), pp. 2691–2702.
3. Rao, G.C., *Almost Distributive Lattices*, Doctoral Thesis, Dept. of Mathematics, Andhra University, Visakhapatnam (1980).
4. Rao, G.C., and S.S. Raju, *Residuated Almost Distributive Lattices – I*, International Journal of Mathematical Archive, Vol. 8, No. 10 (2017), pp. 143–152.
5. Rao, G.C., and S.S. Raju, *Residuated Almost Distributive Lattices – II*, International Journal of Mathematical Archive, Vol. 8, No. 10 (2017), pp. 190–198.
6. Swamy, U.M., and G.C. Rao, *Almost Distributive Lattices,* J. Aust. Math. Soc. (Series A), Vol. 31 (1981), pp. 77–91.
7. Ward, M., and R.P. Dilworth, *Residuated Lattices,* Proceedings of the National Academy of Sciences, Vol. 24 (1938), pp. 162–164.
8. Ward, M., and R.P. Dilworth, *Residuated Lattices*, Transactions of the American Mathematical Society, Vol. 45 (1939), pp. 335–354.